First-order differential equations

Classify and solve the following first-order differential equations:

1. $3x^2y^2 + 2x^3yy' = 0$
2. $y' + y = xy^3$
3. $dy/dx = \frac{e^x - y}{x}$

Solution

1. The given differential equation:

$$3x^2y^2 + 2x^3yy' = 0$$

is an **exact differential equation**. This is because it can be expressed in exact differential form, where there exists a function $F(x, y)$ such that:

$$\frac{\partial F}{\partial x} = M(x, y), \quad \frac{\partial F}{\partial y} = N(x, y)$$

and the differential equation can be written as:

$$M(x, y)\, dx + N(x, y)\, dy = 0$$

In our case, we can rewrite the equation as follows:

1. Multiply both sides by dx:

$$3x^2y^2\, dx + 2x^3yy'\, dx = 0$$

2. Observing that $y'\, dx = dy$, we have:

$$3x^2y^2\, dx + 2x^3y\, dy = 0$$

3. Reordering the terms:

$$(3x^2y^2)\, dx + (2x^3y)\, dy = 0$$

Now we identify:

$$M(x, y) = 3x^2y^2, \quad N(x, y) = 2x^3y$$

To verify if the equation is exact, we check if:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

We compute the partial derivatives:
- $\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(3x^2y^2) = 6x^2y$
- $\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(2x^3y) = 6x^2y$

Integrating both functions:

$$\int (3x^2y^2)\, dx = x^3y^2 + C_1$$

$$\int (2x^3y)\, dy = x^3y^2 + C_2$$

The final result:

$$x^3y^2 = C$$

2. This is a Bernoulli differential equation. We divide the expression by y^3:

$$y'y^{-3} + y^{-2} = x$$

We set $z = y^{-2}$, then $z' = -2y^{-3}y'$, $z'/-2 = y^{-3}y'$, so the equation becomes:

$$\frac{-z'}{2} + z = x$$

$$z' + -2z = -2x$$

This is a linear equation. We propose the substitution $z = uv$, with $z' = u'v + v'u$:

$$u'v + v'u - 2(uv) = -2x$$
$$v(u' - 2u) + v'u = -2x$$

We solve separately:

$$u' - 2u = 0$$
$$v'u = -2x$$

From the first equation:

$$\frac{du}{dx} - 2u = 0$$
$$\frac{du}{u} = 2\,dx$$
$$\ln(u) = 2x$$
$$u = e^{2x}$$

Inserting into the second equation:

$$v'e^{2x} = -2x$$
$$dv = -2xe^{-2x}\,dx$$

We solve the integral on both sides:

$$\int -2xe^{-2x}\,dx$$

Using integration by parts: $\int fg' = fg - \int f'g$,

$$f = -2x, \quad g' = e^{-2x}$$

$$f' = -2, \quad g = \frac{e^{-2x}}{-2}.$$

$$\int -2xe^{-2x}\,dx = -2x\left(\frac{e^{-2x}}{-2}\right) - \int -2\left(\frac{e^{-2x}}{-2}\right) = xe^{-2x} + \frac{e^{-2x}}{2} + C = e^{-2x}\left(x + \frac{1}{2}\right) + C$$

$$v = e^{-2x}\left(x + \frac{1}{2}\right) + C$$

Thus, recall that $z = uv = e^{2x}\left(e^{-2x}\left(x + \frac{1}{2}\right) + C\right) = x + \frac{1}{2} + Ce^{2x}$. Since $z = y^{-2}$, we have:

$$y^{-2} = x + \frac{1}{2} + Ce^{2x}$$

$$y = \left(x + \frac{1}{2} + Ce^{2x}\right)^{-1/2}$$

3. $dy/dx = \frac{e^x - y}{x}$

Rearranging:

$$x\,dy + (y - e^x)\,dx = 0$$

It holds that $M'_x = 1 = N'_y = 1$, where $M = x$, $N = e^x - y$. We integrate:

$$\int x\,dy = xy$$

$$\int (y - e^x)\,dx = yx - e^x$$

Thus:

$$yx - e^x = C$$

$$y(x) = \frac{e^x + C}{x}$$